

A note on the Lee-Yang singularity coupled to 2d quantum gravity

J. Ambjørn^{a,b}, A. Görlich^{a,c}, A.C. Ipsen^a and H.-G. Zhang^c

^a The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
email: ambjorn@nbi.dk, goerlich@nbi.dk, asgercro@nbi.dk

^b Radboud University Nijmegen
Institute for Mathematics, Astrophysics and Particle Physics (IMAPP),
Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands.

^c Institute of Physics, Jagiellonian University,
Reymonta 4, PL 30-059 Krakow, Poland.
email: zhang@th.if.uj.edu.pl

Abstract

We show how to obtain the critical exponent of magnetization in the Lee-Yang edge singularity model coupled to two-dimensional quantum gravity.

1 Introduction

Two-dimensional quantum Liouville gravity and the theory of random triangulations (or matrix models) most likely describe the same theory, two-dimensional quantum gravity coupled to conformal field theories with a central charge $c \leq 1$. The two realizations are sufficiently different that the “proof” that they describe the same theory is basically by comparing the result of calculations of certain “observables”. The major problem of such a comparison has been to identify the observables to be compared in the two formulations. This problem has to a large extent been solved in [2] for one and two-point correlation functions and in [3] for three- and four-points correlation functions. Here we will address an observable, the so-called “magnetization” at the Lee-Yang edge singularity. We will show how the general assumptions of operator mixing put forward in [1, 2, 3] allow us to obtain agreement between the critical exponent of the Lee-Yang “magnetization” calculated in quantum Liouville gravity and using matrix models.

The rest of this article is organized as follows: in the next section we recapture how to calculate the magnetization exponent σ in the Ising model and at the Lee-Yang edge singularity using standard conformal field theory. In sec. 3 we then show how to reconcile Liouville and matrix model results.

2 Ising models and dimer models

The Ising model on an arbitrary connected graph G_V with V vertices and L links is defined by

$$Z_{G_V}(\beta, H) = \sum_{\{\sigma_i\}} \exp \left(\beta \sum_{\langle ij \rangle=1}^L \sigma_i \sigma_j + H \sum_{i=1}^V \sigma_i \right), \quad (1)$$

where the Ising spin σ_i (which can take values ± 1) is located at vertex i , $\langle ij \rangle$ symbolizes that vertices i and j are neighbors in G_V , and β and H signify inverse the temperature and a magnetic field, respectively.

If G_V is a regular two-dimensional lattice, e.g. a square lattice, the partition function $Z_{G_V}(\beta, H = 0)$ has a second order phase transition for a certain value β_c in the limit $V \rightarrow \infty$. Let us calculate

$$\langle e^{H \sum_i \sigma_i} \rangle_{\beta=\beta_c, H=0} = e^{-F_{G_V}(H)}, \quad (2)$$

using the partition function $Z_{G_V}(\beta_c, 0)$. For large V the free energy $F_{G_V}(H)$ becomes extensive and the magnetization m is given by

$$F_{G_V}(H) = f(H) V (1 + o(V)), \quad m = -\frac{df}{dH} \sim |H|^\sigma, \quad \sigma = \frac{1}{15}, \quad (3)$$

for small H .

The two-dimensional Ising model at its critical point β_c is a conformal field theory with central charge $c = 1/2$. Let us recall how the above result is derived using conformal field theory. Consider a conformal field theory and let Φ be a primary operator with scaling dimension Δ_0 , i.e. $\Phi(\sqrt{\lambda}x) = \lambda^{-\Delta_0}\Phi(x)$ (we consider Φ to be the product of its holomorphic and anti-holomorphic parts, i.e. real). Under a scaling $x \rightarrow \sqrt{\lambda}x$ we thus have

$$A = \int d^2x \rightarrow \lambda A, \quad D_0 = \int d^2x \Phi(x) \rightarrow \lambda^{1-\Delta_0} D_0. \quad (4)$$

We can study a “deformation” away from the conformal point by adding the term

$$\delta D_0 = \delta \int d^2x \Phi, \quad [\delta] = [A]^{\Delta_0-1} \quad (5)$$

to the action. The last equation in (5) states the dimension of the coupling constant δ in terms of the dimension of the area A of the 2d universe. As in eq. (2) we can write

$$\langle e^{-\delta D_0} \rangle_0 = e^{-F_A(\delta)}, \quad (6)$$

where the average is calculated at the critical point. For large areas A we expect F_A to be extensive. For dimensional reasons we thus have, δ being the only coupling constant,

$$F_A(\delta) = f(\delta)A(1 + o(A)), \quad f(\delta) = k \delta^{\frac{1}{1-\Delta_0}}. \quad (7)$$

The “ Φ magnetization” is thus

$$m_\Phi = -\frac{df}{d\delta} \sim \delta^{\Delta_0/(1-\Delta_0)}, \quad \text{i.e. } \sigma_\Phi = \frac{\Delta_0}{1-\Delta_0}. \quad (8)$$

Applying this to the spin operator $\Phi_{1,2}$ of the (3,4) minimal conformal field theory which has central charge $c = 1/2$ and corresponds to the Ising model, we have $\Delta_0 = 1/16$ and thus $\sigma_{\Phi_{1,2}} = 1/15$ in agreement with (3). For the (2,5) minimal conformal field theory which has $c = -22/5$ there is only one non-trivial primary operator, again $\Phi_{1,2}$, and $\Delta_0 = -1/5$. The corresponding magnetization exponent is $\sigma_{\Phi_{1,2}} = -1/6$.

Everything said above can be directly transferred to quantum Liouville gravity as long as we consider the partition function for a fixed area which we then take large to avoid finite size effects. More precisely, the partition function for a conformal field theory with central charge c coupled to the Liouville field and with the area of the 2d “universe” fixed to be A is defined as

$$Z_A = \int \mathcal{D}\varphi \mathcal{D}\psi e^{-S_L(\varphi, \hat{g}) - S_c(\psi, \hat{g})} \delta \left(\int d^2x \sqrt{\hat{g}} e^{\alpha\varphi} - A \right). \quad (9)$$

In (9) $S_c(\psi)$ is the matter action and $S_L(\varphi)$ the Liouville action. \hat{g}_{ab} is a fiducial metric in the decomposition of the metric $g_{ab} = e^\varphi \hat{g}_{ab}$, thereby defining the Liouville field. Changing variables $\varphi \rightarrow \varphi + \rho$ in the functional integral allow us to obtain (for surfaces with spherical topology)

$$Z_A \sim A^{\gamma_0-3}, \quad \gamma_0 = \frac{c-1-\sqrt{(25-c)(1-c)}}{12}. \quad (10)$$

For a given conformal field theory and a given primary field Φ , the observable D_0 defined above and the area A are changed to

$$D = \int d^2x \sqrt{\hat{g}} e^{\beta\varphi} \Phi, \quad A = \int d^2x \sqrt{\hat{g}} e^{\alpha\varphi} I \quad (11)$$

In particular the area has become an observable on equal footing with D , associated with the (trivial) primary field I (the identity). The coefficients β, α are determined by the requirement that the observables D and A are invariant under diffeomorphisms and in 2d this implies that they are invariant under conformal transformations [11]. However, D still has a scaling dimension relative to the area A . Let us define the expectation value of an observable \mathcal{O} for fixed area as

$$\langle \mathcal{O} \rangle_A = \frac{1}{Z_A} \int \mathcal{D}\varphi \mathcal{D}\psi \mathcal{O} e^{-S_L(\varphi, \hat{g}) - S_c(\psi, \hat{g})} \delta \left(\int d^2x \sqrt{\hat{g}} e^{\alpha\varphi} - A \right). \quad (12)$$

One has

$$\langle f(\lambda^{-\beta/\alpha} D) \rangle_{\lambda A} = \langle f(D) \rangle_A \quad (13)$$

for any function f . This follows by the change of integration variable $\varphi \rightarrow \varphi + \alpha^{-1} \log \lambda$ in the functional integral (12). In particular we have

$$\langle D \rangle_{\lambda A} = \lambda^{\beta/\alpha} \langle D \rangle_A, \quad \text{i.e.} \quad 1 - \Delta = \frac{\beta}{\alpha}, \quad (14)$$

by analogy with (4). The scaling dimension Δ is thus determined by α and β and is given by the KPZ formula [10]

$$\Delta = \frac{\sqrt{1-c+24\Delta_0} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}. \quad (15)$$

As in the ordinary conformal field theory case we can define the “magnetization” related to Φ by considering the perturbation away from the conformal point by the action

$$\delta D = \delta \int d^2x \sqrt{\hat{g}} e^{\beta\varphi} \Phi, \quad [\delta] = [A]^{\Delta-1}, \quad (16)$$

in analogy with (5). As in (6) we have

$$\langle e^{-\delta D} \rangle_A = e^{-F_A(\delta)}, \quad F(\delta) = f(\delta)A(1 + G(A)), \quad f(\delta) = k \delta^{1/(1-\Delta)}. \quad (17)$$

The “magnetization” is thus

$$m = -\frac{df}{d\delta} \sim \delta^{\Delta/(1-\Delta)}, \quad \text{i.e.} \quad \sigma = \frac{\Delta}{1-\Delta}. \quad (18)$$

In the case of the Ising model (i.e. $c = 1/2$) coupled to the Liouville field the exponent Δ_0 changes from $1/16$ to $\Delta = 1/6$ according to (16). Thus we find that σ_0 changes from $1/15$ to $\sigma = 1/5$. This value was first obtained using the random matrix models in [6] and is a strong test of the equivalence between the continuum limit of the random surface models coupled to matter and quantum Liouville gravity. Applied to the (2,5) minimal conformal field theory coupled to the Liouville field, σ_0 changes from $-1/6$ to $\sigma = -1/3$.

Finally it can be convenient to consider the grand partition function where the area is not kept fixed

$$Z(\mu, \delta) = \int dA \, Z_A \, e^{-\mu A} \langle e^{-\delta D} \rangle_A \sim \left(\mu + k \delta^{1/(1-\Delta)} \right)^{2-\gamma_0}. \quad (19)$$

We obtain

$$Z(\mu, 0) \sim \mu^{2-\gamma_0}, \quad Z(0, \delta) = \delta^{2-\gamma(\delta)}, \quad \gamma(\delta) = \frac{\gamma_0 - 2\Delta}{1-\Delta}. \quad (20)$$

We also observe that if the action $\mu A + \delta D$ is viewed as a small perturbation away from the conformal point $\mu = \delta = 0$ and μ and δ are of the same order of magnitude, the singular behavior of $Z(\mu, \delta)$ is dominated by $\mu^{(2-\gamma_0)}$ if the scaling dimension $\Delta > 0$. If $\Delta < 0$, as can be the case for non-unitary conformal field theories, the singular behavior of $Z(\mu, \delta)$ will be dominated by $\delta^{(2-\gamma(\delta))}$. We note for future reference that for the (2,5) minimal conformal field theory $\gamma_0 = -3/2$ and $\gamma(\delta) = -1/3$. In a grand canonical context it is natural to define

$$Z(\mu, \delta) = e^{-F(\mu, \delta)}, \quad \langle A \rangle_{\mu, \delta} = -\frac{dF}{d\mu}, \quad M(\delta) = -\frac{dF}{d\delta} = m(\delta) \langle A \rangle_{\mu, \delta}, \quad (21)$$

and we have

$$\langle A \rangle_{\mu, \delta} = \frac{1}{\mu + k \delta^{1/(1-\Delta)}}, \quad m(\delta) \sim \delta^{\Delta/(1-\Delta)}. \quad (22)$$

For a given value of δ we have

$$\langle A \rangle_{\mu, \delta} \rightarrow \infty \quad \text{for} \quad \mu \searrow \bar{\mu}(\delta), \quad (23)$$

where the condition

$$\bar{\mu}(\delta) + k\delta^{1/(1-\Delta)} = 0 \quad (24)$$

determines the “critical” value of the cosmological constant μ for a given value of δ . In particular we have

$$\frac{d\bar{\mu}}{d\delta} \sim m(\delta). \quad (25)$$

2.1 Dimers

Consider the Ising model on the graph G_V defined above. It has a high temperature expansion

$$Z_{G_V} = (2 \cosh H)^V (\cosh \beta)^L \times \left[1 + \tanh^2 H [\theta(1)\beta + O(\beta^2)] + \tanh^4 H [\theta(2)\beta^2 + O(\beta^4)] + \dots \right] \quad (26)$$

where $\theta(n)$ is the number of ways one can put down n dimers on the graph G_V without the dimers touching each other (so-called hard dimers). For imaginary magnetic fields it is thus possible to take the high temperature limit where $\beta \rightarrow 0$ and $H = i\tilde{H} \rightarrow i\pi/2$ in such a way that $\xi = \beta \tanh^2 H$ is kept fixed. In this limit the terms in the bracket $[\dots]$ in eq. (26) become the partition function

$$Z_{G_V}(\xi) = \sum_n \theta(n) \xi^n, \quad (\xi = -\beta \tan^2 \tilde{H}) \quad (27)$$

of a hard dimer model with fugacity ξ (which is negative for $\tilde{H} \in]0, \pi/2[$). For $\beta < \beta_c$ the Ising model is known to have a phase transition at a critical, purely imaginary magnetic field $H_c(\beta) = i\tilde{H}_c(\beta)$, the so-called Lee-Yang edge singularity [7] (assuming as before that we have a regular graph G_V , and that we take $V \rightarrow \infty$). It is also known that one can formally associate a “magnetization” to this transition [8]:

$$Z_{G_V}(\beta, \tilde{H}) = e^{-F_{G_V}(\beta, \tilde{H})}, \quad F_{G_V}(\beta, \tilde{H}) \sim f(\beta, \tilde{H}) V, \quad (28)$$

where

$$m(\beta) = -\frac{df}{d(\Delta\tilde{H})} \sim (\Delta\tilde{H})^{\sigma_0}, \quad \Delta\tilde{H} = \tilde{H} - \tilde{H}_c(\beta). \quad (29)$$

The critical exponent σ_0 is independent of β for $\beta < \beta_c$. $\tilde{H}_c(\beta) \rightarrow \pi/2$ for $\beta \rightarrow 0$ and at this point we can extract σ from the dimer partition function (27). The dimer model has a critical point ξ_c for a negative value of the fugacity ξ which is precisely the limit of $-\beta \tan^2 \tilde{H}(\beta)$ for $\beta \rightarrow 0$. Writing

$$Z_{G_V}(\xi) = e^{-F_{G_V}(\xi)}, \quad F_{G_V}(\xi) = f(\xi) V, \quad (30)$$

we obtain

$$m = -\frac{df}{d\Delta\xi} \sim (\Delta\xi)^{\sigma_0}, \quad \Delta\xi = \xi - \xi_c. \quad (31)$$

Finally it was shown in [9] that the critical behavior of the Lee-Yang edge singularity or the hard dimer model could be associated with the (2,5) minimal conformal field theory, and from the above arguments, using conformal field theory we know the corresponding $\sigma_0 = -1/6$. This is in agreement with numerical determinations of σ_0 on regular lattices.

Once this is established we can formally couple the Lee-Yang edge singularity to quantum gravity in the sense that the critical behavior is determined by the coupling between the (2,5) conformal field theory and the Liouville theory. From the above we thus expect the magnetization exponent to change from $-1/6$ to $-1/3$, and we would naively expect to obtain that result if we could explicitly solve the Ising model in an imaginary magnetic field or the hard dimer model on the set of random graphs used to represent 2d gravity. In fact one can solve both models on random graphs and one obtains $\sigma = 1/2$ [4].

3 Operator mixing

Let us for simplicity choose to work with the dimer model and discuss how we can re-interpret the result of [4] using the general philosophy outlined in [1, 2, 3]. The coupling of the dimer model to 2d gravity is done by summing over connected random graphs G_V . Here we restrict ourselves to a set of planar graphs, i.e. we define

$$Z_V(\xi) = \sum_{G_V} \frac{1}{C_{G_V}} Z_{G_V}(\xi), \quad (32)$$

where C_G denotes the order of the automorphism group of the graph G . We can introduce a grand partition function by also summing over graphs with different number of vertices:

$$Z(g, \xi) = \sum_V g^V Z_V(\xi). \quad (33)$$

Let us choose the simplest set of planar random graphs, namely the set where all vertices have order four. The corresponding $Z(g, \xi)$ can be calculated using matrix model techniques [12, 4]. For details we refer to [4]. Here we are only interested in the result. There exists a critical ξ_c . For each $\xi \geq \xi_c$ there exists a corresponding critical $\bar{g}(\xi)$, the radius of convergence of the power series (33). We write

$$Z_V(\xi) = e^{-F_V(\xi)}, \quad F_V(\xi) = f(\xi)V(1 + o(V)), \quad \log \bar{g}(\xi) = f(\xi). \quad (34)$$

On a regular lattice one would clearly identify $f(\xi)$ as the free energy density and expect to calculate the critical exponent σ according to (31). This calculation was performed in [4]:

$$\bar{g}(\xi) = \frac{1}{450\xi^2} \left[(1 + 10\xi)^{3/2} - 1 \right] - \frac{1}{30\xi} \quad (35)$$

i.e. expanding around $\xi_c = 1/9$ one obtains

$$\Delta\bar{g}(\xi) + \frac{10}{9}\Delta\xi = \frac{20\sqrt{10}}{9}\Delta\xi^{3/2} + O(\Delta\xi^2), \quad (36)$$

where

$$\Delta\xi = \xi - \xi_c, \quad \Delta\bar{g}(\xi) = \bar{g}(\xi) - \bar{g}(\xi_c). \quad (37)$$

Differentiating (36) after $\Delta\xi$ we obtain

$$\left. \frac{d\bar{g}}{d\xi} \right|_{\text{singular}} = \left. \frac{d \log \bar{g}}{d\xi} \right|_{\text{singular}} \sim \Delta\xi^{1/2}. \quad (38)$$

Clearly this is at odds with the KPZ value $\sigma = -1/3$ mentioned above for the Lee-Yang edge singularity. We now explain how this is due to operator mixing of A and D , following the logic outlined in [1, 2, 3].

Denote $\bar{g}(\xi_c)$ by g_c . The first observation is that [12, 4]

$$Z(g, \xi_c) \Big|_{\text{singular}} = \Delta g^{-1/3-2}, \quad \Delta g = g_c - g, \quad (39)$$

i.e. one obtains $\gamma(\delta)$ ($= -1/3$) rather than γ_0 ($= -3/2$) for the critical susceptibility exponent related Z . Naively one would have made the following identification in (33)

$$\left(\frac{g}{g_c} \right)^V \rightarrow e^{-\mu A} \quad (40)$$

by introducing a scaling parameter a (with the dimension of length relative to A which we define to have the dimension of length squared)

$$\Delta g = \mu a^2, \quad A = V a^2, \quad a \rightarrow 0. \quad (41)$$

But this is clearly too simple as it would imply a critical behavior $\Delta g^{-\gamma_0-2}$ in (39) according to Liouville theory. Δg has to contain some reference to the coupling δ . In some sense this is natural since both A and D appear when we move away from the conformal point $\mu = \delta = 0$. Fixing $\xi = \xi_c$ and changing $g_c \rightarrow g_c - \Delta g$ is one way to move away from the point g_c, ξ_c corresponding to $\mu = \delta = 0$. The change (36) is another way, where we move along the critical

line with a $\Delta\bar{g}(\xi)$ determined by $\Delta\xi$. It should thus be compared to (24) where $\bar{\mu}(\delta) + k\delta^{1/(1-\Delta)} = 0$, which defines “criticality” in the theory perturbed by the A, D terms in the action. This condition allows us to obtain the relation between $\mu a^2, \delta a^3$ and $\Delta g, \Delta\xi$ if we, in accordance with [1, 2, 3], assume that we deal with an analytic coupling constant redefinition. To lowest order, which is all we need, we thus have

$$a^2 \mu = \Delta g(\xi) + c_2 \Delta\xi, \quad a^3 \delta = c_3 \Delta g(\xi) + \Delta\xi. \quad (42)$$

The condition $\bar{\mu} + k\delta^{2/3} = 0$ implies

$$\Delta\bar{g}(\xi) + c_3^{-1} \Delta\xi = c_3^{-1} \left(k^{-1}(c_3^{-1} - c_2) \right)^{3/2} \Delta\xi^{3/2} + O(\Delta\xi^2). \quad (43)$$

Comparing with (36) we obtain

$$a^3 \delta = \Delta\xi + \frac{9}{10} \Delta\bar{g}(\xi), \quad a^2 \bar{\mu}(\delta) = \Delta\bar{g}(\xi) + d \Delta\xi, \quad (44)$$

where $d = 10/9 - k(2\sqrt{10})^{2/3}$. This shows explicitly that Δg couples to δ as anticipated from eq. (39).

By construction we now have $\bar{\mu}(\delta) \sim \delta^{2/3}$ and thus the correct Liouville magnetization. Further, it is amusing to check how the “wrong” result (38) actually becomes correct if one pays attention to the details¹. (38) is obtained by differentiating (36) after $\Delta\xi$. For the special linear combination (44) eq. (36) can be written as

$$a^3 \delta(\Delta\xi, \Delta\bar{g}(\xi)) = \frac{20\sqrt{10}}{9} \Delta\xi^{3/2} + O(\Delta\xi^2), \quad (45)$$

and differentiating with respect to $\Delta\xi$ leads to

$$a^3 \frac{d\delta}{d\Delta\xi} \sim \Delta\xi^{1/2} \quad \text{or} \quad \frac{d\delta}{d\bar{\mu}} \sim \bar{\mu}^{1/2} + O(a), \quad (46)$$

i.e. according to eq. (25) exactly the correct Liouville equation for the magnetization m if $\sigma = -1/3$.

As mentioned one can also solve the Ising model coupled to 2d gravity [5, 6]. The matrix models use the grand canonical ensemble of graphs, i.e. starting with the partition function (1) one performs the same steps as in eqs. (32) and (33) for the dimer model. We thus have a partition function $Z(g, \beta, H)$. Above the critical temperature we find a critical line with a critical imaginary magnetic field

¹The author of [4] had no motivation to pay attention to details, since his work was done before the understanding of the possibility of operator mixing. In fact his seminal paper was precisely what eventually led to this understanding.

[4] $H_c(\beta) = i\tilde{H}_c(\beta)$, $\beta < \beta_c$, analogous to what we find on a fixed graph. For a fixed value of $\beta < \beta_c$ we have an equation similar to the dimer equation (36) [4]

$$\Delta\bar{g}(\tilde{H}) + d_3\Delta\tilde{H} \sim \Delta\tilde{H}^{3/2}, \quad \Delta\tilde{H} = \tilde{H} - \tilde{H}_c(\beta), \quad (47)$$

from which one would conclude that $\sigma = 1/2$. As for the dimer model, this should be understood as the result of operator mixing, and one should really write

$$a^2\bar{\mu} = \Delta\bar{g}(\tilde{H}) + d_2\Delta\tilde{H}, \quad a^3\delta = d_3^{-1}\Delta\bar{g}(\tilde{H}) + \Delta\tilde{H} \quad (48)$$

in order to recover the KPZ exponent.

Let us briefly mention the ordinary critical point of the Ising model on a dynamical graph. The critical exponents calculated in [5, 6] match the KPZ results, even without accounting for mixing. Regarding σ (and γ_0) one can explicitly check that the naive calculation is unaffected by operator mixing (cf. the discussion after (20)). When the magnetic field is zero the model has a \mathbb{Z}_2 symmetry, which guarantees that the spin operator $\Phi_{1,2}$ is not turned on in the continuum language. This, in turn, ensures that also the exponent α associated with the thermal operator $\Phi_{2,1}$ comes out “right” in [6].

4 Discussion

We have shown how the calculation in [4] leads to agreement between the critical exponents of the “magnetization” calculated in the hard dimer model coupled to dynamical triangulations and in quantum Liouville theory coupled to a (2,5) minimal conformal field theory. The price of this agreement is that the naive separation between geometric and matter degrees of freedom which might seem self-evident for models of spins living on dynamical graphs can thus not be taken for granted.

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